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EQUIVALENT SPATIAL AND MATERIAL DESCRIPTIONS OF FINITE DEFORMATION ELASTOPLASTICITY IN PRINCIPAL AXES

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Abstract—A spatial description of the theory of rate-independent finite deformation elastoplasticity, in which the stress tensor is defined through the strain energy function, is discussed. The main assumption of isotropic elastic response and invariance requirements under superposed rigid body motion restrict the acceptable forms of the strain energy function to those given in terms of principal values of the strain measure of elastic distortion. The formulation is developed on a manifold, and the corresponding material description is obtained simply by pull-back of the derived spatial form, by appealing to the notion of covariance. The method of principal axes is systematically exploited to derive the explicit expression for the stress tensor computation for an arbitrary form of the strain energy function, and the explicit form of the evolution equation for an arbitrary form of the yield function. A model problem of volume-preserving plastic flow is discussed in the closure.

NOTATION

Objects defined on the reference configuration $\mathscr{B}$:

- $\mathbf{G}$ covariant metric tensor in the reference (undeformed) configuration
- $\mathbf{E}$ covariant Green–Lagrange finite strain tensor
- $\mathbf{C}$ covariant right Cauchy–Green deformation tensor
- $\mathbf{S}$ contravariant second Piola–Kirchhoff stress tensor
- $\mathbf{C}^{\mathrm{p}}$ covariant right Cauchy–Green plastic deformation tensor
- $\mathbf{n}_i$ principal directions in the reference configuration

Objects defined on the current (deformed) configuration $\varphi(\mathscr{B})$:

- $\mathbf{g}$ covariant metric tensor in the current configuration
- $\boldsymbol{\tau}$ contravariant Kirchhoff stress tensor
- $\mathbf{b}^{\mathrm{e}}$ contravariant left Cauchy–Green elastic deformation tensor
- $\boldsymbol{v}_i$ principal directions in the current configuration

Two point tensors:

- $\mathbf{F}$ deformation gradient (mixed tensor)
- $\mathbf{F}^{\mathrm{e}}\mathbf{F}^{\mathrm{p}}$ multiplicative decomposition of $\mathbf{F}$

Scalars:

- λ_i^{e} principal elastic stretches
- τ_i principal stresses
- ξ internal variable which controls hardening
- q thermodynamically conjugate variable to ξ
- ψ strain energy function
- ϕ yield function

1. INTRODUCTION

A sound theoretical foundation of the micro-mechanics of elastoplastic crystals has been provided by Hill and Havner (1982) and Asaro (1983), among others. The same level of success has not yet been attained with the phenomenological models of finite deformation elastoplasticity, and not a single model has been met with a universal acceptance and applicability to a most general situation. Many issues still appear to be unsettled, for example: invariance requirements under the superposed rigid body motion [e.g. see Casey and Naghdi (1980) and Dashner (1986)], the role of the plastic spin [e.g. see Mandel (1973) and Dafalias (1985)], equivalence of different decompositions of the strain rate [e.g. see Lee

(1980) and Nemat-Nasser (1982)], equivalence of the Eulerian and Lagrangian descriptions [e.g. see Casey and Naghdi (1988)] etc.

In this work we discuss a form of the general theory of rate-independent finite elastoplasticity in which most of the conflicting issues, alluded to in the above, can be either settled or circumvented altogether. The main restriction, under which this can be worked out, implies an *isotropic* elastic response. For most of the metals and alloys [e.g. see Drucker (1988)] the isotropy of the elastic response of the lattice structure can be considered as a quite realistic assumption. Notice, however, that the assumption of this kind does not preclude an induced anisotropy of the instantaneous tangent modulus in the presence of plastic flow.

If one is to assume that the strain energy depends only on the elastic distortion, i.e. change in local geometry between the current and the *unstressed* configuration, the isotropy restriction imposes a strain energy form given in terms of invariants of the elastic strain measure. A particularly elegant form of the finite plasticity formulation of this kind can be given in the space of principal axes with the strain energy considered as a function of the principal *elastic* stretches. It is interesting to note that the principal axis methodology was also favored in a fundamental work of Hill (1978). While several recent works on numerical implementation of finite deformation elastoplasticity [e.g. see Weber and Anand (1990), Eterovic and Bathe (1990), Perić and Owen (1992) and Simo (1992)] also discuss the formulations in terms of principal stretches, a typical development is restricted to the classical three-dimensional Euclidean space.

In this work, however, we take those considerations to a more general setting of differential manifolds [e.g. see Lang (1985) and Marsden and Hughes (1983)]. For a number of important applications, such as the finite deformation analysis of elastoplastic membrane shells [e.g. see Ibrahimbegović (1993)], this is indeed the only appropriate framework. In addition, a general formulation set on a manifold can clarify with particular ease (simply by means of pull-back and push-forward) different possibilities for equivalent descriptions of finite deformation plasticity. In this work, we focus on two choices, Eulerian† or spatial and Langrangian or material descriptions, which are also most commonly selected for description of finite deformation elasticity [e.g. see Truesdell and Noll (1965), pp. 37–39]. In passing we comment on how this development relates to the commonly selected description of finite deformation plasticity set in intermediate configuration.

Working within the framework of manifolds, one can furthermore obtain a finite plasticity equivalent [e.g. see Simo (1988) and Moran *et al.* (1990)] of the notion of the covariant constitutive theory, which was first presented for finite elasticity in Marsden and Hughes (1983). The preference therein [see Marsden and Hughes (1983), Simo (1988) and Moran *et al.* (1990)] was given to the invariant forms of the strain energy in terms of invariants of the chosen strain measure, and a potential development based on principal stretches is judged to be too complicated [see Marsden and Hughes (1983), p. 220]. We hope to show in this paper that the method of principal axes and the strain energy forms set in terms of principal elastic stretches, provide indeed a very suitable framework for the development of a covariant theory of finite deformation elastoplasticity. Moreover, as discussed in Ibrahimbegović (1993), the developments of this kind provide a very suitable basis for the numerical implementation, as already recognized within a simpler Euclidean framework (e.g. Weber and Anand (1990), Eterovic and Bathe (1991), Perić and Owen (1992) and Simo (1992)].

2. SPATIAL DESCRIPTION: BASIC FORMULATION

We assume that the multiplicative decomposition of the deformation gradient **F** gives rise to a stress-free intermediate configuration‡ [e.g. see Lee (1969), Mandel (1973) and Lubliner (1980)], i.e.

† Memory effect of plastic process requires that the entire particle motion be traced by the material integration, and spatial objects become functions of reference coordinates [e.g. see Argyris *et al.* (1980)], so that the spatial description of this kind is not a strict Eulerian technique.

‡ Multiplicative decomposition of the deformation gradient is defined only point-wise, so that the intermediate "configuration" does not necessarily represent a collection of compatible neighborhoods.

$$\mathbf{F} = \mathbf{F}^{e}\mathbf{F}^{p}, \tag{1}$$

where $\mathbf{F}^{e}$ is the Fréchet derivative of elastic deformation from the intermediate to the current configuration, while $\mathbf{F}^{p}$ is the Fréchet derivative of plastic deformation from the reference to the intermediate configuration.

We will first formulate the theory in the spatial description. The elastic strain measure which reflects elastic distortion of the lattice structure, can be defined in terms of the left Cauchy–Green elastic deformation tensor $\mathbf{b}^{e}$, by using the decomposition in eqn (1) as

$$\mathbf{b}^{e} = \mathbf{F}^{e}\bar{\mathbf{G}}^{-1}\mathbf{F}^{eT}, \tag{2}$$

where $\bar{\mathbf{G}}$ is the metric tensor in the intermediate configuration. As elaborated upon in what follows, we assume that the value of stress tensor is defined through the strain energy function

$$\psi = \psi(\mathbf{b}^{e}, \mathbf{g}, \xi), \tag{3}$$

where $\mathbf{g}$ is the metric tensor in the current configuration, and ξ is an internal variable which controls hardening. For simplicity, we assume that ξ is a *scalar*, which implies the isotropic hardening.† Note that with a slight abuse of notation, in eqn (3) and what follows, we denote the function and its value with the same symbol.

The invariance requirements under rigid body motion superposed on the spatial configuration, along with the isotropy requirements in the reference configuration, lead to the strain energy form in terms of the principal values of $\mathbf{b}^{e}$, or alternatively, in terms of elastic principal stretches λ_i^{e}, i.e.

$$\psi = \psi(\lambda_i^{e}, \xi). \tag{4}$$

The elastic principal stretches are the solutions to the eigenvalue problem

$$[\mathbf{b}^{e} - (\lambda_i^{e})^2\mathbf{g}^{-1}]\,\boldsymbol{v}_i = \mathbf{0}. \tag{5}$$

The eigenvalue problem in eqn (5) above is set in terms of contravariant tensors, so that the principal vectors $\boldsymbol{v}_i$ are covariant vectors, or one-forms [see Marsden and Hughes (1983), p. 49]. Principal stretches λ_i^{e}, on the other hand, are independent of a particular choice of coordinates for the eigenvalue problem in eqn (5) [e.g. see Green and Zerna (1968), p. 24].

An arbitrary coordinate representation of eqn (5) leads to a general linear eigenvalue problem for the matrix pencil of two symmetric matrices $(\mathbf{b}^{e}, \mathbf{g}^{-1})$. Recall [e.g. see Parlett (1980), p. 307] that the positive definiteness of $\mathbf{g}$ is the guarantee for all the λ_i^{e} to be real. Also recall [e.g. see Parlett (1980)] the orthogonality property of the eigenvectors $\boldsymbol{v}_i$ as

$$\boldsymbol{v}_i \cdot \mathbf{g}^{-1}\boldsymbol{v}_j = \delta_{ij}, \tag{6}$$

where δ_{ij} is the Kronecker symbol, and $\cdot$ denotes the scalar product. The orthogonality property of eigenvectors in eqn (6) plays an important role in the developments which follow. For example, by using the orthogonality property in eqn (6), we can obtain the spectral decomposition of $\mathbf{b}^{e}$ as

$$\mathbf{g}\mathbf{b}^{e} = \sum_i (\lambda_i^{e})^2 \boldsymbol{v}_i \otimes \mathbf{g}^{-1}\boldsymbol{v}_i \Rightarrow \mathbf{b}^{e} = \sum_i (\lambda_i^{e})^2 \mathbf{g}^{-1}\boldsymbol{v}_i \otimes \mathbf{g}^{-1}\boldsymbol{v}_i, \tag{7}$$

where $\otimes$ denotes the tensor product.

† Note that the choice of isotropic hardening refutes the questions of whether the intermediate configuration is physically attainable.

It is important to note that the isotropy implies that the energy-conjugate Kirchhoff stress tensor $\boldsymbol{\tau}$ shares the same eigenvectors $\boldsymbol{v}_i$, i.e. we have

$$[\boldsymbol{\tau}-\tau_i\mathbf{g}^{-1}]\boldsymbol{v}_i=\mathbf{0}\Rightarrow\boldsymbol{\tau}=\sum_i\tau_i\mathbf{g}^{-1}\boldsymbol{v}_i\otimes\mathbf{g}^{-1}\boldsymbol{v}_i, \tag{8}$$

where τ_i are the principal values of Kirchhoff stress.

Another important ingredient of the plasticity theory discussed herein is the choice of the yield function in terms of Kirchhoff stress. The invariance requirements under rigid body motion superposed on the spatial configuration limit the acceptable forms of the yield function to those given in terms of invariants of $\boldsymbol{\tau}$, or in terms of its principal values τ_i, i.e.

$$\phi=\phi(\boldsymbol{\tau},\mathbf{g},q)=\phi(\tau_i,q). \tag{9}$$

In eqn (9), q denotes the variable which is thermo-dynamically conjugate to the hardening variable ξ. Both q and $\boldsymbol{\tau}$ are specified through the corresponding stress–strain relations, as elaborated upon shortly.

Having defined eqns (1)–(9) as the basic ingredients of the theory, we show that the principle of maximum plastic dissipation [e.g. see Lubliner (1984) or Simo (1988)] along with the standard thermodynamic considerations (Coleman and Gurtin, 1967) are the only things we need in order to provide the remaining ingredients of the theory: stress–strain constitutive relations for $\boldsymbol{\tau}$ and q, and evolution equations for $\mathbf{b}^e$ and ξ, together with the loading–unloading conditions.

Consider the spatial form of the dissipation inequality for the isothermal case

$$\mathscr{D}:=\boldsymbol{\tau}\cdot\mathbf{d}-\frac{\mathrm{d}}{\mathrm{d}t}\psi(\mathbf{b}^e,\mathbf{g},\xi)\geqslant 0. \tag{10}$$

In eqn (10) above, $\mathbf{d}$ is the spatial rate-of-deformation tensor [e.g. see Marsden and Hughes (1983), p. 98], which can be computed by means of the Lie derivative [e.g. see Lang (1985) p. 109] of the spatial metric $\mathbf{g}$ as

$$\begin{aligned}\mathbf{d}&=\tfrac{1}{2}L_v[\mathbf{g}]\\&=\tfrac{1}{2}\mathbf{F}^{-\mathrm{T}}\left\{\frac{\partial}{\partial t}[\mathbf{F}^{\mathrm{T}}\mathbf{g}\mathbf{F}]\right\}\mathbf{F}^{-1}\\&=\tfrac{1}{2}\{\dot{\mathbf{g}}+\mathbf{l}^{\mathrm{T}}\mathbf{g}+\mathbf{g}\mathbf{l}\},\end{aligned} \tag{11}$$

where $\mathbf{l}=\dot{\mathbf{F}}\mathbf{F}^{-1}$ is the spatial velocity gradient.

The material time derivative of the strain energy in eqn (10) can be computed by applying the chain rule as

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(\mathbf{b}^e,\mathbf{g},\xi)=\frac{\partial\psi}{\partial\mathbf{b}^e}\cdot L_v[\mathbf{b}^e]+\frac{\partial\psi}{\partial\mathbf{g}}\cdot L_v[\mathbf{g}]+\frac{\partial\psi}{\partial\xi}\dot{\xi}. \tag{12}$$

In eqn (12) above, $L_v[\mathbf{b}^e]$ is the Lie derivative of the elastic left Cauchy–Green tensor, which can be computed as

$$\begin{aligned}L_v[\mathbf{b}^e]&=\mathbf{F}\left\{\frac{\partial}{\partial t}[\mathbf{F}^{-1}\mathbf{b}^e\mathbf{F}^{-\mathrm{T}}]\right\}\mathbf{F}^{\mathrm{T}}\\&=\dot{\mathbf{b}}^e+\mathbf{l}\mathbf{b}^e+\mathbf{b}^e\mathbf{l}^{\mathrm{T}}.\end{aligned} \tag{13}$$

In order to work out the expression in eqn (12), the key result needed is given in the following.

Proposition 1

$$\frac{\partial \psi}{\partial \mathbf{b}^e} = \mathbf{g}\frac{\partial \psi}{\partial \mathbf{g}}\mathbf{b}^{e-1}. \tag{14}$$

Proof: By using the invariant form of ψ, we can compute

$$\frac{\partial \psi}{\partial \mathbf{b}^e} = \sum_i \frac{\partial \psi}{\partial \lambda_i^e}\frac{\partial \lambda_i^e}{\partial \mathbf{b}^e}, \tag{15}$$

where $\partial\psi/\partial\lambda_i^e$ can be computed directly from eqn (4). Partial derivatives of the principal stretches λ_i^e with respect to $\mathbf{b}^e$ can be computed from eqn (5) in an intrinsic form [e.g. see Truesdell and Noll (1965), p. 25] as

$$\mathrm{d}\mathbf{b}^e\mathbf{v}_i - 2\lambda_i^e\left(\frac{\partial \lambda_i^e}{\partial \mathbf{b}^e}\cdot \mathrm{d}\mathbf{b}^e\right)\mathbf{g}^{-1}\mathbf{v}_i + [\mathbf{b}^e - (\lambda_i^e)^2\mathbf{g}^{-1}]\,\mathrm{d}\mathbf{v}_i = \mathbf{0}. \tag{16}$$

Scalar-multiplication of eqn (16) above by the vector $\mathbf{v}_i$, along with the eigenvector orthogonality property in eqn (6) and the eigenvalue identity in (5), leads to

$$(\mathrm{d}\mathbf{b}^e\mathbf{v}_i)\cdot\mathbf{v}_i - 2\lambda_i^e\left(\frac{\partial \lambda_i^e}{\partial \mathbf{b}^e}\cdot \mathrm{d}\mathbf{b}^e\right) = 0, \tag{17}$$

from where it follows that

$$\frac{\partial \lambda_i^e}{\partial \mathbf{b}^e} = \frac{1}{2\lambda_i^e}\mathbf{v}_i \otimes \mathbf{v}_i. \tag{18}$$

Similarly, using the invariant form of the strain energy, we can write

$$\frac{\partial \psi}{\partial \mathbf{g}} = \sum_i \frac{\partial \psi}{\partial \lambda_i^e}\frac{\partial \lambda_i^e}{\partial \mathbf{g}}, \tag{19}$$

where $\partial\lambda_i^e/\partial\mathbf{g}$ can be computed from eqn (5) as

$$\mathrm{d}\mathbf{g}\,\mathbf{b}^e\mathbf{v}_i - 2\lambda_i^e\left(\frac{\partial \lambda_i^e}{\partial \mathbf{g}}\cdot \mathrm{d}\mathbf{g}\right)\mathbf{v}_i + [\mathbf{g}\mathbf{b}^e - (\lambda_i^e)^2\mathbf{1}]\,\mathrm{d}\mathbf{v}_i = \mathbf{0}. \tag{20}$$

Upon scalar-multiplication with the vector $\mathbf{g}^{-1}\mathbf{v}_i$ we can rewrite eqn (20), using the orthogonality property in eqn (6) and eigenvalue statement in (5), as

$$(\mathrm{d}\mathbf{g}\,\mathbf{b}^e\mathbf{v}_i)\cdot\mathbf{g}^{-1}\mathbf{v}_i - 2\lambda_i^e\left(\frac{\partial \lambda_i^e}{\partial \mathbf{g}}\cdot \mathrm{d}\mathbf{g}\right) = 0, \tag{21}$$

from where it follows that

$$\frac{\partial \lambda_i^e}{\partial \mathbf{g}} = \frac{1}{2\lambda_i^e}\mathbf{g}^{-1}\mathbf{v}_i \otimes \mathbf{b}^e\mathbf{v}_i. \tag{22}$$

Finally, comparing eqns (15) and (18) on one side, against (19) and (22) on the other, we recover the result in eqn (14). ■

Having obtained the results in eqns (11)–(14) we can rewrite the dissipation inequality (10) as

$$\mathscr{D} := \left(\boldsymbol{\tau} - 2\frac{\partial \psi}{\partial \mathbf{g}}\right) \cdot \mathbf{d} - \left(\mathbf{g}\,\frac{\partial \psi}{\partial \mathbf{g}}\,\mathbf{b}^{e-1}\right) \cdot L_v[\mathbf{b}^e] - \frac{\partial \psi}{\partial \xi}\dot{\xi} \geqslant 0. \tag{23}$$

Assuming that the elastic process is non-dissipative [e.g. see Lubliner (1980)], from eqn (23) above we get the stress–strain relationship in terms of

$$\boldsymbol{\tau} = 2\frac{\partial \psi}{\partial \mathbf{g}}, \tag{24}$$

as the equivalent of the Doyle–Ericksen formula [e.g. see Doyle and Ericksen (1956)] for the case of finite deformation elastoplasticity.

Note that, upon substituting the results in eqns (19) and (22) into (24) above and using the eigenvalue identity in eqn (5), we can explicitly compute the stress tensor according to

$$\begin{aligned} \boldsymbol{\tau} &= \sum_i \frac{1}{\lambda_i^e}\frac{\partial \psi}{\partial \lambda_i^e}\,\mathbf{g}^{-1}\boldsymbol{v}_i \otimes \mathbf{b}^e\boldsymbol{v}_i \\ &= \sum_i \lambda_i^e \frac{\partial \psi}{\partial \lambda_i^e}\,\mathbf{g}^{-1}\boldsymbol{v}_i \otimes \mathbf{g}^{-1}\boldsymbol{v}_i, \end{aligned} \tag{25}$$

which, along with the result in eqn (8), results in $\tau_i = \lambda_i^e\,\partial\psi/\partial\lambda_i^e$.

If we denote as q the variable which is thermodynamically conjugate to the hardening variable ξ, i.e.

$$q = -\frac{\partial \psi}{\partial \xi}, \tag{26}$$

we can then obtain a reduced form of the dissipation inequality as

$$\mathscr{D}^p := \boldsymbol{\tau} \cdot \left(-\tfrac{1}{2}\mathbf{g}L_v[\mathbf{b}^e]\mathbf{b}^{e-1}\right) + q\dot{\xi} \geqslant 0, \tag{27}$$

where $\mathscr{D}^p$ is the plastic dissipation.

As shown in Lubliner (1984) and Simo (1988), the principle of maximum plastic dissipation can be used in order to come up with the evolution equations and loading–unloading conditions. The basic idea is to look among all the admissible states $(\boldsymbol{\tau}^*, \mathbf{q}^*)$ which satisfy the yield condition $\phi(\mathbf{t}^*, \mathbf{g}, q^*) \leqslant 0$, and find the state which renders the plastic dissipation in eqn (27) maximum. This can be interpreted as a constrained minimization problem, and formulated with the aid of the Lagrange multiplier procedure as

$$\mathscr{L}^p(\boldsymbol{\tau}^*, q^*, \dot{\gamma}) := -\mathscr{D}^p(\boldsymbol{\tau}^*, q^*) + \dot{\gamma}\phi(\boldsymbol{\tau}^*, \mathbf{g}, q^*), \tag{28}$$

where $\dot{\gamma} \geqslant 0$ is the Lagrange multiplier. The Kuhn–Tucker optimality conditions [e.g. see Luenberger (1984), p. 314, Strang (1986), p. 724] of the Lagrangian in eqn (28) above lead to the evolution equations for $\mathbf{b}^e$ and ξ

$$L_v[\mathbf{b}^e] = 2\dot{\gamma}\mathbf{g}^{-1}\,\frac{\partial\phi}{\partial\boldsymbol{\tau}}\mathbf{b}^e, \tag{29}$$

$$\dot{\xi} = \dot{\gamma}\frac{\partial\phi}{\partial q} \tag{30}$$

and loading–unloading conditions

$$\dot{\gamma} \geqslant 0\,;\, \phi(\boldsymbol{\tau}, \mathbf{g}, q) \leqslant 0\,;\, \dot{\gamma}\phi = 0. \tag{31}$$

In closing this section, we remark that one can also compute an explicit form of eqn (29) by making use of the following proposition.

Proposition 2

$$\frac{\partial\phi}{\partial\boldsymbol{\tau}} = \sum_i \frac{\partial\phi}{\partial\tau_i}\boldsymbol{v}_i \otimes \boldsymbol{v}_i. \tag{32}$$

Proof: By using the chain rule on the invariant form of the yield condition in eqn (9), we can state

$$\frac{\partial\phi}{\partial\boldsymbol{\tau}} = \sum_i \frac{\partial\phi}{\partial\tau_i}\frac{\partial\tau_i}{\partial\boldsymbol{\tau}}. \tag{33}$$

In addition, from the eigenvalue identity in eqn (8) it follows that

$$\mathrm{d}\boldsymbol{\tau}\boldsymbol{v}_i - \left(\frac{\partial\tau_i}{\partial\boldsymbol{\tau}}\cdot\mathrm{d}\boldsymbol{\tau}\right)\mathbf{g}^{-1}\boldsymbol{v}_i + [\boldsymbol{\tau} - \tau_i\mathbf{g}^{-1}]\,\mathrm{d}\boldsymbol{v}_i = \mathbf{0}, \tag{34}$$

which, upon scalar-multiplication with the vector $\boldsymbol{v}_i$ and use of the eigenvector orthogonality and eigenvalue identity, leads to

$$\frac{\partial\tau_i}{\partial\boldsymbol{\tau}} = \mathbf{v}_i \otimes \boldsymbol{v}_i. \tag{35}$$

Substituting eqn (35) into (33) leads to (32). ■

3. MATERIAL DESCRIPTION: COVARIANT FORM OF THE THEORY

Having completed the developments of the spatial description of the theory in the previous section, in this section we proceed to obtain an equivalent material description by appealing to the notion of covariance [see Marsden and Hughes (1983), p. 199]. Namely, we impose that a covariant constitutive theory should remain invariant not only under a rigid body motion superposed on the current configuration (which entails no change of metric), but also under a superposed diffeomorphism $\boldsymbol{\alpha}$, as long as the new metric is computed properly by means of push-forward. Assuming that the superposed diffeomorphism is equal to the inverse of the actual motion, i.e. $\boldsymbol{\alpha} \equiv \boldsymbol{\varphi}^{-1}$, the push-forward by $\boldsymbol{\alpha}$ will correspond to the pull-back by φ [e.g. see Lang (1985), p. 106]. In particular, we would have

$$\begin{aligned}\boldsymbol{\alpha}_*[\mathbf{g}^{-1}] &= \boldsymbol{\varphi}^*[\mathbf{g}^{-1}] \\ &= \mathbf{F}^{-1}\mathbf{g}^{-1}\mathbf{F}^{-\mathrm{T}} \\ &= \mathbf{C}^{-1}\end{aligned} \tag{36}$$

and

$$\begin{aligned}\alpha_*[\mathbf{b}^e] &= \varphi^*[\mathbf{b}^e] \\ &= \mathbf{F}^{-1}\mathbf{b}^e\mathbf{F}^{-T} \\ &= \mathbf{F}^{p-1}\mathbf{G}^{-1}\mathbf{F}^{p-T} \\ &= \mathbf{C}^{p-1}.\end{aligned} \tag{37}$$

Note that under these transformations the solutions for the eigenvalues λ_i^e remain preserved, since

$$\begin{aligned}\mathbf{0} &= [\mathbf{b}^e - (\lambda_i^e)^2\mathbf{g}^{-1}]\mathbf{v}_i \\ &= [\mathbf{F}^{-1}\mathbf{b}^e\mathbf{F}^{-T} - (\lambda_i^e)^2\mathbf{F}^{-1}\mathbf{g}^{-1}\mathbf{F}^{-T}]\mathbf{F}^T\mathbf{v}_i \\ &= [\mathbf{C}^{p-1} - (\lambda_i^e)^2\mathbf{C}^{-1}]\mathbf{n}_i.\end{aligned} \tag{38}$$

The transformation in eqn (38) above gives rise to the material form of the eigenvectors

$$\mathbf{n}_i = \mathbf{F}^T\mathbf{v}_i, \tag{39}$$

with the orthogonality property $\mathbf{n}_i \cdot \mathbf{C}^{-1}\mathbf{n}_j = \delta_{ij}$, which reveals the role of the right Cauchy–Green strain tensor $\mathbf{C}$ as the induced metric in the reference configuration.

The strain energy form in eqn (4) also remains preserved, for it depends only on λ_i^e. Taking into account that λ_i^e are also the solutions to the eigenvalue problem in eqn (38), we can write an alternative expression for the strain energy as

$$\psi = \psi(\mathbf{C}^p, \mathbf{C}, \xi). \tag{40}$$

Similarly, the pull-back of the Kirchhoff stress tensor τ gives rise to the second Piola–Kirchhoff stress tensor $\mathbf{S}$ as

$$\begin{aligned}\alpha_*[\tau] &= \varphi^*[\tau] \\ &= \mathbf{F}^{-1}\tau\mathbf{F}^{-T} \\ &= \mathbf{S},\end{aligned} \tag{41}$$

which shares the same eigenvectors $\mathbf{n}_i$ as in eqn (39), since

$$\begin{aligned}\mathbf{0} &= [\tau - \tau_i\mathbf{g}^{-1}]\mathbf{v}_i \\ &= [\mathbf{F}^{-1}\tau\mathbf{T}^{-T} - \tau_i\mathbf{T}^{-1}\mathbf{g}^{-1}\mathbf{F}^{-T}]\mathbf{F}^T\mathbf{v}_i \\ &= [\mathbf{S} - \tau_i\mathbf{C}^{-1}]\mathbf{n}_i.\end{aligned} \tag{42}$$

Also note from eqn (42) that $\mathbf{S}$ has the same eigenvalues τ_i as in eqn (8). This implies that the invariant form of the yield criterion in eqn (9) remains preserved. Alternatively, we can also write the yield criterion in terms of material objects as

$$\phi(\mathbf{S}, \mathbf{C}, q) = 0. \tag{43}$$

The pull-back of the expression for computing the Kirchoff stress in eqn (25), with the help of eqns (39) and (42), leads to

$$\begin{aligned}\mathbf{S} &= 2\mathbf{F}^{-1}\frac{\partial \psi}{\partial \mathbf{g}}\mathbf{F}^{-\mathrm{T}} \\ &= 2\mathbf{F}^{-1}\mathbf{g}^{-1}\mathbf{F}^{-\mathrm{T}}\left[\sum_i \tau_i \mathbf{n}_i \otimes \mathbf{n}_i\right]\mathbf{F}^{-1}\mathbf{g}^{-1}\mathbf{F}^{-\mathrm{T}} \\ &= 2\sum_i \tau_i \mathbf{C}^{-1}\mathbf{n}_i \otimes \mathbf{C}^{-1}\mathbf{n}_i \\ &= 2\frac{\partial \psi}{\partial \mathbf{C}}. \end{aligned} \tag{44}$$

Finally, note that the evolution equation for ξ remains the same as in eqn (30). The remaining evolution equation in (29), however, is transformed by using the results in eqns (1) and (13) and the definition in (2) to get

$$\begin{aligned}\alpha_*[L_v[\mathbf{b}^e]] &= \varphi^*[L_v[\mathbf{b}^e]] \\ &= \mathbf{F}^{-1}L_v[\mathbf{b}^e]\mathbf{F}^{-\mathrm{T}} \\ &= \frac{\partial}{\partial t}[\mathbf{F}^{-1}\mathbf{b}^e\mathbf{F}^{-\mathrm{T}}] \\ &= \frac{\partial}{\partial t}[\mathbf{F}^{\mathrm{p}-1}\bar{\mathbf{G}}^{-1}\mathbf{F}^{\mathrm{p}-\mathrm{T}}] \\ &= \dot{\mathbf{C}}^{\mathrm{p}-1}, \end{aligned} \tag{45}$$

where we denoted $\mathbf{C}^{\mathrm{p}} = \mathbf{F}^{\mathrm{p}\mathrm{T}}\bar{\mathbf{G}}\mathbf{F}^{\mathrm{p}}$ the right Cauchy–Green plastic deformation tensor. In addition, using the results in eqns (32), (39) and (42) to get

$$\begin{aligned}\alpha_*\left[\mathbf{g}^{-1}\frac{\partial \phi}{\partial \boldsymbol{\tau}}\mathbf{b}^e\right] &= \varphi^*\left[\mathbf{g}^{-1}\frac{\partial \phi}{\partial \boldsymbol{\tau}}\mathbf{b}^e\right] \\ &= \mathbf{F}^{-1}\mathbf{g}^{-1}\mathbf{F}^{-\mathrm{T}}\left[\sum_i \frac{\partial \phi}{\partial \tau_i}\mathbf{n}_i \otimes \mathbf{n}_i\right]\mathbf{F}^{-1}\mathbf{b}^e\mathbf{F}^{-\mathrm{T}} \\ &= \mathbf{C}^{-1}\frac{\partial \phi}{\partial \mathbf{S}}\mathbf{C}^{\mathrm{p}-1}, \end{aligned} \tag{46}$$

we arrive at the material form of eqn (29) given as

$$\dot{\mathbf{C}}^{\mathrm{p}-1} = -2\dot{\gamma}\mathbf{C}^{-1}\frac{\partial \phi}{\partial \mathbf{S}}\mathbf{C}^{\mathrm{p}-1}, \tag{47}$$

with

$$\frac{\partial \phi}{\partial \mathbf{S}} = \sum_i \frac{\partial \phi}{\partial \tau_i}\mathbf{n}_i \otimes \mathbf{n}_i$$

as the material form of eqn (32).

Remark 1

If the spatial diffeomorphism superposed on the current configuration is assumed to be equal to the inverse of the motion from the intermediate to the current configuration, we will obtain the form of the finite deformation theory set in the intermediate configuration. For example, we would have

$$\mathbf{F}^{e-1}\mathbf{b}^{e}\mathbf{F}^{e-T} = \bar{\mathbf{G}}^{-1}$$

$$\mathbf{F}^{e-1}\mathbf{g}^{-1}\mathbf{F}^{e-T} = \bar{\mathbf{C}}^{e-1}$$

$$\mathbf{F}^{e-1}\boldsymbol{\tau}\mathbf{F}^{e-T} = \bar{\mathbf{S}}, \tag{48}$$

which would again preserve the invariant form of the presented theory. In the vast majority of recent developments on numerical implementation of the finite deformation elasto-plasticity [e.g. see Moran *et al.* (1990), Weber and Anand (1990), Eterovic and Bathe (1990), Perić and Owen (1992), Cutino and Oritz (1992)] the description associated with the intermediate configuration was indeed a preferred choice. However, under present restriction of the isotropic elastic response, one can see that the choice of configuration is immaterial. ■

Remark 2

The state variables of the finite elastoplasticity formulation are either $\{\mathbf{C}, \mathbf{C}^{p}, \xi\}$ in Lagrangian or $\{\mathbf{b}^{e}, \mathbf{g}, \xi\}$ in Eulerian description. Other variables, such as $\boldsymbol{\tau}$, $\mathbf{S}$ or q, are dependent on the state variables. The central problem of computational plasticity [e.g. see Hughes (1983)] is to trace the time histories of the state variables through an incremental sequence. The crucial simplification of such a problem is provided by the operator split method [e.g. see Simo and Ortiz (1985), Simo (1988)], where the state variable computation is divided into the computation of either $\mathbf{C}$ in Lagrangian or $\mathbf{g}$ in Eulerian description, which sets the new current configuration, and the computation of the remaining state variables (either $\{\mathbf{C}^{p}, \xi\}$ or $\{\mathbf{b}^{e}, \xi\}$), which corresponds to the intermediate configuration update. In a three-dimensional case or a two-dimensional planar problem, the spatial description provides the possibilities to simplify numerical implementation by using the standard Euclidean metric [e.g. see Simo (1992)]. However, for a more general two-dimensional subset of the three-dimensional case, such as space-curved membrane shells, material description is more suitable for numerical implementation [see Ibrahimbegović (1993)]. ■

Remark 3

Even if we select a special set of orthogonal material coordinates, in which the coordinate representation of the metric tensor is the identity matrix, i.e. even $\mathbf{G} = \mathbf{I}$, it is still not convenient to choose the Green–Lagrange strain tensors $\{\mathbf{E}, \mathbf{E}^{p}\}$ for the state variables instead of Cauchy–Green strains $\{\mathbf{C}^{p}, \mathbf{C}\}$, as suggested in the works of Green and Naghdi (1965) or Naghdi (1990). The inconvenience is brought about by a more involved form of the evolution equation, and resulting difficulties which would complicate the description of the problem in terms of the principal values of $\{\mathbf{E}, \mathbf{E}^{p}\}$. ■

4. MODEL PROBLEM: ISOCHORIC PLASTIC FLOW

In this section we discuss a particular choice for the set of constitutive equations which fits within the general framework discussed in the preceding sections. In particular, two ingredients of the theory, the yield function ϕ and the strain energy ψ, which are earlier left in a general form are to be specified now.

For the majority of metals [e.g. see Drucker (1988)], the well-known yield criterion of von Mises is used most often. This yield criterion can be stated in terms of invariants of the deviatoric part of the Kirchhoff stress as

$$\phi(\boldsymbol{\tau}, \mathbf{g}, q) := \tfrac{1}{2}\mathrm{dev}_{g}[\boldsymbol{\tau}] \cdot \mathrm{dev}_{g}[\boldsymbol{\tau}] - \tfrac{1}{3}(\tau_{y} - q)^{2} = 0, \tag{49}$$

where τ_{y} is a uniaxial yield stress, and $\mathrm{dev}_{g}[\bullet]$ denotes the deviatoric part of the tensor, i.e.

$$\mathrm{dev}_g[\boldsymbol{\tau}] = \boldsymbol{\tau} - \tfrac{1}{3}(\boldsymbol{\tau}\cdot\mathbf{g})\mathbf{g}^{-1}. \tag{50}$$

By substituting eqn (50) into (49), we can get an alternative form of the yield criterion as

$$\phi(\boldsymbol{\tau}, \mathbf{g}, q) := \tfrac{1}{2}[\mathbf{g}\boldsymbol{\tau}\cdot\mathbf{g}\boldsymbol{\tau} - \tfrac{1}{3}(\boldsymbol{\tau}\cdot\mathbf{g})^2] - \tfrac{1}{3}(\tau_y - q)^2 = 0. \tag{51}$$

Note that in eqn (51) one can also write $\boldsymbol{\tau}\cdot\mathbf{g} \equiv (\mathbf{g}\boldsymbol{\tau})\cdot\mathbf{1}$. Furthermore, from eqns (8) and (42) it follows that the corresponding principal axis representations of $\mathbf{g}\boldsymbol{\tau}$ and $\mathbf{CS}$ are the same, which directly leads to the equivalent material form of the yield criterion in eqn (51) as

$$\phi(\mathbf{S}, \mathbf{C}, q) := \tfrac{1}{2}[\mathbf{CS}\cdot\mathbf{CS} - \tfrac{1}{3}(\mathbf{S}\cdot\mathbf{C})^2] - \tfrac{1}{3}(\tau_y - q)^2 = 0. \tag{52}$$

In the principal axis representations of either spatial or material form of the von Mises yield criterion, we recover the well-known form [e.g. see Lubliner (1990), p. 129] given as

$$\phi(\tau_i, q) := \tfrac{1}{3}(\tau_1^2 + \tau_2^2 + \tau_3^2 - \tau_1\tau_2 - \tau_2\tau_3 - \tau_3\tau_1) - \tfrac{1}{3}(\tau_y - q)^2 = 0. \tag{53}$$

One consequence of the yield criterion in eqn (53), or any pressure-insensitive yield criterion, is the exact preservation of plastic volume, or plastic incompressibility condition. If we denote by $J := \det \mathbf{F}$, $J^e = \det \mathbf{F}^e$ and $J^p = \det \mathbf{F}^p$, which form eqn (1) results with $J = J^e J^p$, then the plastic incompressibility condition can be stated as

$$J^p = 1 \Rightarrow J \equiv J^e. \tag{54}$$

In order to confirm the result in eqn (54) above, we consider the material time derivative of $J^e = \lambda_1^e \lambda_2^e \lambda_3^e$ given as

$$\dot{J}^e = \frac{\partial J^e}{\partial \mathbf{b}_e} \cdot L_v[\mathbf{b}^e] + \frac{\partial J^e}{\partial \mathbf{g}} \cdot L_v[\mathbf{g}], \tag{55}$$

where with the aid of eqn (18) we have

$$\begin{aligned} \frac{\partial J^e}{\partial \mathbf{b}^e} &= \sum_i \frac{\partial J^e}{\partial \lambda_i^e} \frac{\partial \lambda_i^e}{\partial \mathbf{b}^e} \\ &= J^e \sum_i \frac{1}{2(\lambda_i^e)^2} \boldsymbol{v}_i \otimes \boldsymbol{v}_i, \end{aligned} \tag{56}$$

and with the result in eqn (22) we get

$$\begin{aligned} \frac{\partial J^e}{\partial \mathbf{g}} &= \sum_i \frac{\partial J^e}{\partial \lambda_i^e} \frac{\partial \lambda_i^e}{\partial \mathbf{g}} \\ &= J^e \sum_i \frac{1}{2(\lambda_i^e)^2} \mathbf{g}^{-1}\boldsymbol{v}_i \otimes \mathbf{b}^e \boldsymbol{v}_i \\ &= \tfrac{1}{2} J^e \sum_i \mathbf{g}^{-1}\boldsymbol{v}_i \otimes \mathbf{g}^{-1}\boldsymbol{v}_i \\ &= \tfrac{1}{2} J^e \mathbf{g}^{-1}. \end{aligned} \tag{57}$$

By substituting results in eqns (56) and (57) into (55), and making use of (11), (29) and (32), we get

$$\dot{J}^e = J^e\left[\sum_i \frac{1}{2(\lambda_i^e)^2}\boldsymbol{v}_i \otimes \boldsymbol{v}_i\right]\cdot\left[\sum_i(-2\dot{\lambda})(\lambda_i^e)^2\frac{\partial\phi}{\partial\tau_i}\mathbf{g}^{-1}\boldsymbol{v}_i \otimes \mathbf{g}^{-1}\boldsymbol{v}_i\right]+\tfrac{1}{2}J^e\mathbf{g}^{-1}\cdot 2\mathbf{d}$$

$$= -J^e\dot{\gamma}\sum_i\frac{\partial\phi}{\partial\tau_i} + J^e\mathbf{g}^{-1}\cdot\mathbf{d}$$

$$= J^e\mathbf{g}^{-1}\cdot\mathbf{d}, \tag{58}$$

where we have used the fact that the pressure-insensitive form of the yield criterion in (53) results with $\Sigma_i\ \partial\phi/\partial\tau_i = 0$. If we compare the result in eqn (58) with the standard result $\dot{J} = J\mathbf{g}^{-1}\cdot\mathbf{d}$, we directly recover the plastic incompressibility condition in (54).

Next we address the issues of a proper form of the strain energy function for isotropic elastoplasticity. First, by following Lubliner (1972), we will assume an additive decomposition of the strain energy function in eqn (4) as

$$\psi(\lambda_i^e, \xi) = \hat{\psi}(\lambda_i^e) + \Xi(\xi). \tag{59}$$

The particular forms for $\Xi(\xi)$ in eqn (59) above, which control description of the hardening behavior, are often selected by curve-fitting of available test results.

On the other hand, the admissible forms of $\hat{\psi}(\lambda_i^e)$ should satisfy certain conditions imposed by finite elasticity. Namely, since the yield criterion in eqn (50) places no restrictions on volumetric elastic strain, they can be quite large. That these extreme strains should be accompanied by an infinite stress is a physically reasonable requirement which is easy to grasp. With the invariant form of the strain energy in eqns (4) or (59), the physical requirement of this kind can be easily represented in terms of mathematical conditions on the form of $\hat{\psi}$, i.e.

$$\hat{\psi}(\lambda_i^e) \to \infty \text{ as } \{\lambda_1^e\lambda_2^e\lambda_3^e\} \to 0^+$$

$$\hat{\psi}(\lambda_i^e) \to \infty \text{ as } \{\lambda_1^e\lambda_2^e\lambda_3^e\} \to \infty. \tag{60}$$

These conditions are intimately related to the notion of poly-convexity [e.g. see Marsden and Hughes (1983), p. 20, Ciarlet (1988), p. 158]. A general form of the strain energy which satisfies these requirements is given for so-called Ogden's material† [e.g. see Ogden (1984), p. 219, or Ciarlet (1988), p. 181] as

$$\hat{\psi}(\lambda_i^e) = \sum_{\alpha_i} a_i(\lambda_1^{e\alpha_i} + \lambda_2^{e\alpha_i} + \lambda_3^{e\alpha_i}) + \sum_{\beta_j} b_j((\lambda_1^e\lambda_2^e)^{\beta_j} + (\lambda_2^e\lambda_3^e)^{\beta_j} + (\lambda_3^e\lambda_1^e)^{\beta_j}) + \Gamma(\lambda_1^e\lambda_2^e\lambda_3^e) + e, \tag{61}$$

where $\Gamma(\bullet)$ is a convex function of its argument, for example,

$$\Gamma(x) = cx^2 - d\ln(x), \tag{62}$$

and constitutive coefficients are restricted to

$$a_i > 0;\ \alpha_i \geqslant 1;\ b_j > 0;\ \beta_j \geqslant 1;\ c > 0;\ d > 0;\ e \in \mathscr{R}. \tag{63}$$

Besides the restrictions given in eqn (63), the constitutive coefficients should be selected in such a way that the standard form of the isotropic elasticity (St Venant–Kirchhoff material) is recovered for very small elastic strains, i.e. for $\lambda_i^e \mapsto 1$. The strain energy function

† This is an analogy with finite elasticity, where Ogden's material is obtained when the total principal stretches λ_i are used rather than the elastic stretches λ_i^e.

which satisfies all the conditions stated above, and yet has a fairly simple form, is given [see Ciarlet (1988), p. 185] as

$$\hat{\psi}(\lambda_i^e) = a(\lambda_1^{e2}+\lambda_2^{e2}+\lambda_3^{e2})+b((\lambda_1^e\lambda_2^e)^2+(\lambda_2^e\lambda_3^e)^2+(\lambda_3^e\lambda_1^e)^2)+c(\lambda_1^e\lambda_2^e\lambda_3^e)^2+d\ln(\lambda_1^e\lambda_2^e\lambda_3^e)+e, \tag{64}$$

with

$$a=\mu+\tfrac{1}{2}\Gamma'(1);\ b=-\tfrac{1}{2}\mu-\tfrac{1}{2}\Gamma'(1)$$
$$c=\tfrac{1}{4}(\Gamma'(1)+\Gamma''(1));\ d=\tfrac{1}{2}(\Gamma''(1)-\Gamma'(1));\ e=-(3a+3b+c), \tag{65}$$

and specific choice for $\Gamma'(1)\in(-\lambda/2-\mu,\ -\mu)$ and $\Gamma''(1)\in(\lambda/2+\mu,\ \lambda+\mu)$, where λ and μ are the Lamé parameters.

Note that the strain energy function in eqn (64) can also be written directly in terms of state variables $(\mathbf{g}, \mathbf{b}^e)$ in spatial, or $(\mathbf{C}^p, \mathbf{C})$ in material description, upon noting that

$$(\mathbf{g}\mathbf{b}^e)\cdot\mathbf{1}=(\mathbf{C}\mathbf{C}^{p-1})\cdot\mathbf{I}=(\lambda_1^e)^2+(\lambda_2^e)^2+(\lambda_3^e)^2, \tag{66}$$

$$(\mathrm{cof}(\mathbf{g}\mathbf{b}^e))\cdot\mathbf{1}=(\mathrm{cof}(\mathbf{C}\mathbf{C}^{p-1}))\cdot\mathbf{I}=(\lambda_1^e\lambda_2^e)^2+(\lambda_2^e\lambda_3^e)^2+(\lambda_3^e\lambda_1^e)^2, \tag{67}$$

and

$$\det(\mathbf{g}\mathbf{b}^e)=\det(\mathbf{C}\mathbf{C}^{p-1})=(\lambda_1^e\lambda_2^e\lambda_3^e)^2, \tag{68}$$

where $\mathrm{cof}(\mathbf{A})=(\det\mathbf{A})\mathbf{A}^{-T}$.

Remark 4

Another useful form of the strain energy, which can meet the requirement on the small elastic strain limit, is obtained from the strain energy of St Venant–Kirchhoff material by simply replacing the Euler–Lagrange strains by logarithmic strains to get

$$\hat{\psi}(\lambda_i^e)=\tfrac{1}{2}\lambda(\ln\lambda_1^e+\ln\lambda_2^e+\lambda_3^e)^2+\mu((\ln\lambda_1^e)^2+(\ln\lambda_2^e)^2+(\ln\lambda_3^e)^2). \tag{69}$$

This form of the strain energy does not satisfy the poly-convexity conditions only for the extreme elastic strains, and it is still a very useful approximation in the moderate strain regime. It is a very popular choice in numerical implementation [e.g. see Weber and Anand [1990), Eterovic and Bathe (1991), Perić and Owen (1992), Simo (1992)], for it results with significant simplifications. Moreover, in the presence of plane stress condition, this form of the strain energy permits an explicit two-dimensional reduction of the general theory discussed herein, which has a very important application in the analysis of finite deformation of elastoplastic membrane shells [e.g. see Ibrahimbegović (1993)].

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